The homogeneous coordinate ring of the quantum projective plane

Masoud Khalkhali and Ali Moatadelro
Department of Mathematics, University of Western Ontario
London, Ontario, Canada

Abstract

We define holomorphic structures on canonical line bundles on the quantum projective plane. The space of holomorphic sections of these line bundles will determine the quantum homogeneous coordinate ring of $\mathbb{C}P_q^2$. We also show that the holomorphic structure of $\mathbb{C}P_q^2$ is naturally represented by a twisted positive Hochschild 4-cocycle.

Contents

1 Introduction

In this paper we continue a study of complex structures on quantum projective spaces that was initiated in [7] for $\mathbb{C}P_q^1$. In the present paper we consider a

natural holomorphic structure on the quantum projective plane $\mathbb{C}P_q^2$ already presented in [6, 5], and define holomorphic structures on canonical quantum line bundles on it. The space of holomorphic sections of these line bundles then will determine the quantum homogeneous coordinate ring of $\mathbb{C}P_q^2$.

In Section 2, we review basic notions of a complex structure on an involutive algebra as well as complex structures on modules and bimodules over such algebra from [7]. In Section 3, we recall the definition of the quantum projective plane $\mathbb{C}P_q^2$ [6], and its canonical line bundles. In Section 4, we introduce a flat $\overline{\partial}$-connection on the bimodules representing canonical quantum line bundles on $\mathbb{C}P_q^2$. We also establish the compatibility of these connections with the natural tensor product of these bimodules. This compatibility is then used to derive the structure of the quantum homogeneous coordinate ring of $\mathbb{C}P_q^2$ as a twisted polynomial algebra in three variables. In Section 5 we extend the results of Section 4 to L^2-functions and L^2-sections.

In the last section, using the complex structure on $\mathbb{C}P_q^2$, we give a formula for a twisted Hochschild 4-cocycle on $\mathcal{A}(\mathbb{C}P_q^2)$ cohomologous to its fundamental cyclic 4-cocycle which is originally defined via its smooth structure. We also show that this cocycle is twisted positive in an appropriate sense [7]. This fits well with the point of view on holomorphic structures in noncommutative geometry advocated in [3, 4].

2 Preliminaries

In this section we review the general setup of noncommutative complex structure on a given $*$-algebra as introduced in [7].

2.1 Noncommutative complex structures

Let $\mathcal{A}$ be a $*$-algebra over $\mathbb{C}$. A *differential $*$-calculus* for $\mathcal{A}$ is a pair $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$, where $\Omega^\bullet(\mathcal{A}) = \bigoplus_{n\geq 0} \Omega^n(\mathcal{A})$ is a graded differential $*$-algebra with $\Omega^0(\mathcal{A}) = \mathcal{A}$. The differential map $\mathrm{d} : \Omega^n(\mathcal{A}) \to \Omega^{n+1}(\mathcal{A})$ satisfying the graded Leibniz rule, $\mathrm{d}(\omega_1\omega_2) = (\mathrm{d}\omega_1)\omega_2 + (-1)^n\omega_1(\mathrm{d}\omega_2)$ and $\mathrm{d}^2 = 0$. The differential also anti commutes with the $*$-structure: $\mathrm{d}(a^*) = -(\mathrm{d}a)^*$.

Definition 2.1. *A complex structure on an algebra $\mathcal{A}$, equipped with a differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$, is a bigraded differential $*$-algebra $\Omega^{(\bullet,\bullet)}(\mathcal{A})$ and two differential maps $\partial : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p+1,q)}(\mathcal{A})$ and $\overline{\partial} : \Omega^{(p,q)}(\mathcal{A}) \to \Omega^{(p,q+1)}(\mathcal{A})$ such that:*

$$\Omega^n(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{(p,q)}(\mathcal{A}), \quad \partial a^* = -(\overline{\partial}a)^*, \quad \mathrm{d} = \partial + \overline{\partial}. \tag{1}$$

Also, the involution $*$ maps $\Omega^{(p,q)}(\mathcal{A})$ to $\Omega^{(q,p)}(\mathcal{A})$.

We will use the simple notation $(\mathcal{A}, \overline{\partial})$ for a complex structure on $\mathcal{A}$.

Definition 2.2. *Let* $(\mathcal{A}, \overline{\partial})$ *be an algebra with a complex structure. The space of holomorphic elements of* $\mathcal{A}$ *is defined as*

$$\mathcal{O}(\mathcal{A}) := Ker\{\overline{\partial} : \mathcal{A} \to \Omega^{(0,1)}(\mathcal{A})\}.$$

2.2 Holomorphic connections

Suppose we are given a differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$. We recall that a connection on a left $\mathcal{A}$-module $\mathcal{E}$ for the differential calculus $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$ is a linear map $\nabla : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ with left Leibniz property:

$$\nabla(a\xi) = a\nabla\xi + \mathrm{d}a \otimes_{\mathcal{A}} \xi, \quad \forall a \in \mathcal{A}, \forall \xi \in \mathcal{E}. \tag{2}$$

By the graded Leibniz rule, i.e.

$$\nabla(\omega\xi) = (-1)^n \omega\nabla\xi + \mathrm{d}\omega \otimes \xi, \quad \forall \omega \in \Omega^n(\mathcal{A}), \forall \xi \in \Omega(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}, \tag{3}$$

this connection can be uniquely extended to a map, which will be denoted again by ∇, $\nabla : \Omega^\bullet(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^{\bullet+1}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$.

The curvature of such a connection is defined by $F_\nabla = \nabla \circ \nabla$. One can show that, F_∇ is an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E})$.

Definition 2.3. *Suppose* $(\mathcal{A}, \overline{\partial})$ *is an algebra with a complex structure. A holomorphic structure on a left* $\mathcal{A}$*-module* $\mathcal{E}$ *with respect to this complex structure is given by a linear map* $\nabla^{\overline{\partial}} : \mathcal{E} \to \Omega^{(0,1)} \otimes_{\mathcal{A}} \mathcal{E}$ *such that*

$$\nabla^{\overline{\partial}}(a\xi) = a\nabla^{\overline{\partial}}\xi + \overline{\partial}a \otimes_{\mathcal{A}} \xi, \quad \forall a \in \mathcal{A}, \forall \xi \in \mathcal{E}, \tag{4}$$

and such that $F_{\nabla^{\overline{\partial}}} = (\nabla^{\overline{\partial}})^2 = 0$.

Such a connection will be called a flat $\overline{\partial}$-connection. In the case which $\mathcal{E}$ is a finitely generated $\mathcal{A}$-module, $(\mathcal{E}, \nabla^{\overline{\partial}})$ will be called a holomorphic vector bundle.

Associated to a flat $\overline{\partial}$-connection, there exists a complex of vector spaces

$$0 \to \mathcal{E} \to \Omega^{(0,1)} \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^{(0,2)} \otimes_{\mathcal{A}} \mathcal{E} \to ... \tag{5}$$

Here $\nabla^{\overline{\partial}}$ is extended to $\Omega^{(0,q)} \otimes_{\mathcal{A}} \mathcal{E}$ by the graded Leibniz rule. The zeroth cohomology group of this complex is called the space of Holomorphic sections of $\mathcal{E}$ and will be denoted by $H^0(\mathcal{E}, \nabla^{\overline{\partial}})$.

2.3 Holomorphic structures on bimodules

Definition 2.4. *Let* $\mathcal{A}$ *be an algebra with a differential calculus* $(\Omega^\bullet(\mathcal{A}), \mathrm{d})$*. A bimodule connection on an* $\mathcal{A}$*-bimodule* $\mathcal{E}$ *is given by a connection* ∇ *which satisfies a left Leibniz rule as in formula (2) and a right* σ*-twisted Leibniz property with respect to a bimodule isomorphism* $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$*. i.e.*

$$\nabla(\xi a) = \nabla\xi a + \sigma(\xi \otimes \mathrm{d}a), \quad \forall \xi \in \mathcal{E}, \forall a \in \mathcal{A}. \tag{6}$$

The tensor product connection of two bimodule connections ∇_1 and ∇_2 on two $\mathcal{A}$-bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$ with respect to the bimodule isomorphisms σ_1 and σ_2 is a map $\nabla : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ defined by

$$\nabla := \nabla_1 \otimes 1 + (\sigma_1 \otimes 1)(1 \otimes \nabla_2).$$

It can be checked that, ∇ has the right σ-twisted property with $\sigma : \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \mathcal{E}_1 \otimes \mathcal{E}_2$ given by $\sigma = (\sigma_1 \otimes 1) \circ (1 \otimes \sigma_2)$.

3 The quantum projective plane $\mathbb{C}P^2_q$

In this section, we recall the definition of the quantum enveloping algebra $U_q(\mathfrak{su}(3))$, the quantum group $\mathcal{A}(SU_q(3))$ and the pairing between them. We also recall the definition of the quantum projective plane $\mathbb{C}P^2_q$ and its canonical quantum line bundles [6].

3.1 The quantum enveloping algebra $U_q(\mathfrak{su}(3))$

Let $0 < q < 1$. We use the following notation

$$[a, b]_q = ab - q^{-1}ba,\ [z] = \frac{q^z - q^{-z}}{q - q^{-1}},\ \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!}\,,$$
$$[j, k, l]! = q^{-(jk+kl+lj)}\,\frac{[j+k+l]!}{[j]![k]![l]!}\,.$$

The Hopf $*$-algebra $U_q(\mathfrak{su}(3))$ as a $*$-algebra is generated by K_i, K_i^{-1}, E_i, F_i, $i = 1, 2$ with $K_i^* = K_i$, $E_i^* = F_i$ subject to the relations

$$\begin{aligned} &[K_i, K_j] = 0, \quad K_i E_i = q E_i K_i, \quad [E_i, F_i] = (q - q^{-1})^{-1}(K_i^2 - K_i^{-2}), \\ &K_i E_j = q^{-1/2} E_j K_i, \quad [E_i, F_j] = 0, \quad i \neq j\,, \end{aligned}$$

and

$$E_i^2 E_j + E_j E_i^2 = (q + q^{-1}) E_i E_j E_i \quad i \neq j\,.$$

Its coproduct, counit and antipode are defined on generators as

$$\begin{aligned} &\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i, \\ &\Delta(K_i) = K_i \otimes K_i, \quad \epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0, \\ &S(K_i) = K_i^{-1}, \quad S(E_i) = -qE_i, \quad S(F_i) = -q^{-1}F_i. \end{aligned}$$

Let $V(n_1, n_2)$ be the irreducible finite dimensional $*$-representation of $U_q(\mathfrak{su}(3))$ [8] with the orthonormal basis $|n_1, n_2, j_1, j_2, m\rangle$, where indices are restricted by

$$j_i = 0, 1, 2, ..., n_i, \quad \frac{1}{2}(j_1 + j_2) - |m| \in \mathbb{N}. \tag{7}$$

The generators of $U_q(\mathfrak{su}(3))$ act on this basis as

$$K_1|n_1,n_2,j_1,j_2,m\rangle = q^m|n_1,n_2,j_1,j_2,m\rangle,$$
$$K_2|n_1,n_2,j_1,j_2,m\rangle = q^{\frac{3}{4}(j_1-j_2)+\frac{1}{2}(n_2-n_1-m)}|n_1,n_2,j_1,j_2,m\rangle,$$
$$E_1|n_1,n_2,j_1,j_2,m\rangle = \sqrt{[\frac{1}{2}(j_1+j_2)-m][\frac{1}{2}(j_1+j_2)+m+1]}$$
$$|n_1,n_2,j_1,j_2,m+1\rangle,$$
$$E_2|n_1,n_2,j_1,j_2,m\rangle = \sqrt{[\frac{1}{2}(j_1+j_2)-m+1]}A_{j_1,j_2}|n_1,n_2,j_1+1,j_2,m-\frac{1}{2}\rangle$$
$$+\sqrt{[\frac{1}{2}(j_1+j_2)+m]}B_{j_1,j_2}|n_1,n_2,j_1,j_2-1,m-\frac{1}{2}\rangle, \tag{8}$$

where

$$A_{j_1,j_2} := \sqrt{\frac{[n_1-j_1][n_2+j_1+2][j_1+1]}{[j_1+j_2+1][j_1+j_2+2]}}, \tag{9}$$

$$B_{j_1,j_2} := \begin{cases} \sqrt{\frac{[n_1+j_2+1][n_2-j_2+1][j_2]}{[j_1+j_2][j_1+j_2+1]}} & \text{if } j_1+j_2 \neq 0, \\ 1 & \text{if } j_1+j_2 = 0. \end{cases} \tag{10}$$

3.2 The quantum group $\mathcal{A}(SU_q(3))$

As a $*$-algebra, $\mathcal{A}(SU_q(3))$ is generated by u^i_j, $i,j=1,2,3$, satisfying the following commutation relations

$$u^i_k u^j_k = q u^j_k u^i_k, \qquad u^k_i u^k_j = q u^k_j u^k_i \quad \forall\, i<j,$$
$$[u^i_l, u^j_k] = 0, \qquad [u^i_k, u^j_l] = (q-q^{-1})u^i_l u^j_k \quad \forall\, i<j,\ k<l,$$

and a cubic relation

$$\sum_{\sigma\in S_3} (-q)^{l(\sigma)} u^1_{\sigma(1)} u^2_{\sigma(2)} u^3_{\sigma(3)} = 1.$$

In the last equation, sum is taken over all permutation σ on three letters and $l(\sigma)$ is the length of σ. The involution $*$ is deafened as

$$(u^i_j)^* := (-q)^{j-i}(u^{k_1}_{l_1}u^{k_2}_{l_2} - q u^{k_1}_{l_2}u^{k_2}_{l_1}), \tag{11}$$

where as an ordered set, $\{k_1,k_2\} = \{1,2,3\}\setminus\{i\}$ and $\{l_1,l_2\} = \{1,2,3\}\setminus\{j\}$. The Hopf algebra structure is given by

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j, \quad \epsilon(u^i_j) = \delta^i_j, \quad S(u^i_j) = (u^j_i)^*.$$

There exists a non-degenerate pairing between Hopf algebras $\mathcal{A}(SU_q(3))$ and $U_q(\mathfrak{su}(3))$, which allows us to define a left and a right action of $U_q(\mathfrak{su}(3))$ on

$\mathcal{A}(SU_q(3))$. These actions make $\mathcal{A}(SU_q(3))$ an $U_q(\mathfrak{su}(3))$-bimodule $*$-algebra. The actions are defined as

$$h \triangleright a = a_{(1)}\langle h, a_{(2)}\rangle, \quad a \triangleleft h = \langle h, a_{(1)}\rangle a_{(2)}.$$

Here we used Sweedler's notation. Left and right actions on generators are given by (see [5])

$$\begin{aligned} &K_i \triangleright u^j_k = q^{\frac{1}{2}(\delta_{i+1,k}-\delta_{i,k})}u^j_k, \quad E_i \triangleright u^j_k = \delta_{i,k}u^j_{i+1}, \quad F_i \triangleright u^j_k = \delta_{i+1,k}u^j_i, \\ &u^j_k \triangleleft K_i = q^{\frac{1}{2}(\delta_{i+1,j}-\delta_{i,j})}u^j_k, \quad u^j_k \triangleleft E_i = \delta_{i+1,j}u^i_k, \quad u^j_k \triangleleft F_i = \delta_{i,j}u^{i+1}_k. \end{aligned} \tag{12}$$

A linear basis of $\mathcal{A}(SU_q(3))$ corresponding to the Peter-Weyl decomposition is given by (see [5, 6])

$$t(n_1,n_2)^{l_1,l_2,k}_{j_1,j_2,m} := X^{n_1,n_2}_{j_1,j_2,m} \triangleright \{(u^1_1)^*\}^{n_1}(u^3_3)^{n_2} \triangleleft (X^{n_1,n_2}_{l_1,l_2,k})^*. \tag{13}$$

where $X^{n_1,n_2}_{j_1,j_2,m}$ is defined as

$$\begin{aligned} &X^{n_1,n_2}_{j_1,j_2,m} := N^{n_1,n_2}_{j_1,j_2,m} \\ &\sum_{k=0}^{n_1-j_1} \frac{q^{-k(j_1+j_2+k+1)}}{[j_1+j_2+k+1]!} \begin{bmatrix} n_1-j_1 \\ k \end{bmatrix} F_1^{1/2(j_1+j_2)-m+k}[F_2,F_1]_q^{n_1-j_1-k}F_2^{j_2+k}. \end{aligned}$$

The coefficients $N^{n_1,n_2}_{j_1,j_2,m}$ are defined by

$$N^{n_1,n_2}_{j_1,j_2,m} = \sqrt{[j_1+j_2+1]}\sqrt{\frac{[\frac{j_1+j_2}{2}+m]![n_2-j_2]![j_1]![n_1+j_2+1]![n_2+j_1+1]!}{[\frac{j_1+j_2}{2}-m]![n_1-j_1]![j_2]![n_1]![n_2]![n_1+n_2+1]!}}.$$

The Peter-Weyl isomorphism $Q : \mathcal{A}(SU_q(3)) \to \bigoplus_{(n_1,n_2)} V(n_1,n_2) \otimes V(n_1,n_2)$ has the following property for all $h \in U_q(\mathfrak{su}(3))$:

$$\begin{aligned} &Q(h \triangleright t(n_1,n_2)^{l_1,l_2,k}_{j_1,j_2,m}) = h|n_1,n_2,j_1,j_2,m\rangle \otimes |n_1,n_2,l_1,l_2,k\rangle, \\ &Q(t(n_1,n_2)^{l_1,l_2,k}_{j_1,j_2,m} \triangleleft h) = |n_1,n_2,j_1,j_2,m\rangle \otimes \theta(h)|n_1,n_2,l_1,l_2,k\rangle, \end{aligned} \tag{14}$$

where $\theta : U_q(\mathfrak{su}(3)) \to U_q(\mathfrak{su}(3))^{op}$ is the Hopf $*$-algebra isomorphism which is defined on generators as

$$\theta(K_i) = K_i,\ \theta(E_i) = F_i,\ \theta(F_i) = E_i,$$

and satisfying $\theta^2 = id$.

We define the quantum projective plane $\mathbb{C}P^2_q$ as a quotient of the 5-dimensional quantum sphere ([6]). By definition

$$\mathcal{A}(S^5_q) := \{a \in \mathcal{A}(SU_q(3)) |\, a \triangleleft h = \epsilon(h)a\,,\ \forall h \in U_q(\mathfrak{su}(2))\}.$$

As a $*$-algebra, $\mathcal{A}(S^5_q)$ is generated by elements $z_j = u^3_j,\ j = 1,2,3$ of $\mathcal{A}(SU_q(3))$. Abstractly, this algebra is the algebra with generators z_i, z_i^* $i = 1,2,3$ and

subject to the following relations

$$
\begin{aligned}
&z_i z_j = q z_j z_i \quad \forall\, i<j, && z_i^* z_j = q z_j z_i^*, \quad \forall\, i \neq j,\\
&[z_1^*, z_1] = 0, && [z_2^*, z_2] = (1-q^2) z_1 z_1^*,\\
&[z_3, z_3] = (1-q^2)(z_1 z_1^* + z_2 z_2^*), && z_1 z_1^* + z_2 z_2^* + z_3 z_3^* = 1.
\end{aligned}
$$

Now we define the algebra $\mathcal{A}(\mathbb{C}P_q^2)$ of the quantum projective plane as a $*$-subalgebra of $\mathcal{A}(S_q^5)$.

$$
\mathcal{A}(\mathbb{C}P_q^2) := \{a \in \mathcal{A}(S_q^5) |\, a \triangleleft K_1 K_2^2 = a\}.
$$

One can show that [6], $\mathcal{A}(S_q^5) \simeq \bigoplus_{(n_1,n_2)\in\mathbb{N}^2} V(n_1,n_2)$ with the basis $t(n_1,n_2)^{\underline{0}}_{\underline{j}}$, where n_1 and n_2 are non-negative integers. Also $\mathcal{A}(\mathbb{C}P_q^2) \simeq \bigoplus_{n\in\mathbb{N}} V(n,n)$ with the basis $t(n,n)^{\underline{0}}_{\underline{j}}$. Here we have used the multi index notation $\underline{j} = j_1, j_2, m$ and indices j_1, j_2, m are restricted by (7).

For any integer N, we define the space of the canonical quantum line bundle $\mathcal{L}_N$ on $\mathbb{C}P_q^2$ by

$$
\mathcal{L}_N := \{a \in \mathcal{A}(S_q^5) : a \triangleleft K_1 K_2^2 = q^N a\}.
$$

These spaces are $\mathcal{A}(\mathbb{C}P_q^2)$-bimodules. One can see that [6],

$$
\mathcal{L}_N = \bigoplus_{n\in\mathbb{N}} V(n, n+N) \quad \text{if } N \geq 0, \text{ and } \mathcal{L}_N = \bigoplus_{n\in\mathbb{N}} V(n-N, n) \quad \text{if } N < 0.
$$

The basis elements are given by $t(n,n+N)^{\underline{0}}_{\underline{j}}$ for $N \geq 0$ and $t(n-N,n)^{\underline{0}}_{\underline{j}}$ for $N < 0$.

4 The complex structure of $\mathbb{C}P_q^2$

There is a complex structure on $\mathbb{C}P_q^2$ defined in [5, 6]. For future use, we give an explicit description of the spaces $\Omega^{(0,0)}$, $\Omega^{(0,1)}$ and $\Omega^{(0,2)}$:

$$
\Omega^{(0,0)} = \mathcal{L}_0 = \mathcal{A}(\mathbb{C}P_q^2), \quad \Omega^{(0,2)} = \mathcal{L}_3,
$$

and as a subspace of $\mathcal{A}(SU_q(3))^2$, $\Omega^{(0,1)}$ contains all pairs (v_+, v_-) such that the following conditions hold

$$
\begin{aligned}
&(v_+,v_-) \triangleleft K_1 K_2^2 = q^{\frac{3}{2}}(v_+,v_-), && (v_+,v_-) \triangleleft K_1 = (q^{\frac{1}{2}} v_+, q^{-\frac{1}{2}} v_-),\\
&(v_+,v_-) \triangleleft F_1 = (0, v_+), && (v_+,v_-) \triangleleft E_1 = (v_-, 0).
\end{aligned}
\tag{15}
$$

The complex structure on $\mathbb{C}P_q^2$ is given by the maps $\partial : \mathcal{A}(\mathbb{C}P_q^2) \to \Omega^{(1,0)}(\mathbb{C}P_q^2)$ and $\overline{\partial} : \mathcal{A}(\mathbb{C}P_q^2) \to \Omega^{(0,1)}(\mathbb{C}P_q^2)$, which (up to multiplicative constants) are $\partial a = (a \triangleleft E_2, a \triangleleft F_2 E_1)^t$, $\overline{\partial} a = (a \triangleleft F_2 F_1, a \triangleleft F_2)^t$.

In this section we identify the space of holomorphic functions on $\mathbb{C}P_q^2$ and holomorphic sections of $\mathcal{L}_N$.

4.1 Holomorphic functions

Proposition 4.1. *There are no non-trivial holomorphic polynomials on* $\mathbb{C}P^2_q$.

Proof. Let $a = \sum_{n,\underline{j}} \lambda_{n,\underline{j}} t(n,n)^{\underline{0}}_{\underline{j}}$. Then $\overline{\partial} a = 0$ implies that $a \triangleleft F_2 = 0$ and $a \triangleleft F_2 F_1 = 0$. A simple computation shows that $a \triangleleft F_2 = \sum \lambda_{n,\underline{j}} \gamma_n t(n,n)^{1,0,-\frac{1}{2}}_{\underline{j}}$, where $\gamma_n = A_{0,0} = \sqrt{\frac{[n][n+2]}{[2]}}$. This can be obtained by (14), (8) and (9) because $E_2|n,n,0,0,0\rangle = A_{0,0}|n,n,1,0,-\frac{1}{2}\rangle$.
Since $\gamma_n = 0$ iff $n = 0$, all coefficients need to be zero except $c_{0,0}$. Note that the action of F_1 does not put more restrictions on the coefficients. This demonstrates that

$$Ker\{\overline{\partial} : \mathcal{A}(\mathbb{C}P^2_q) \to \Omega^{(0,1)}(\mathbb{C}P^2_q)\} = \langle t(0,0)^{\underline{0}}_{\underline{0}}\rangle = \mathbb{C}.$$

□

This preposition, already has been proved in [6] as a result of a Hodge decomposition.

4.2 Canonical line bundles

Like [5], we define the connection ∇_N on $\mathcal{L}_N$ by $\nabla_N := q^{-N}\Psi_N^\dagger \mathrm{d}\Psi_N$, where Ψ_N is the column vector with components $\psi^N_{i,j,k}$ given by

$$(\psi^N_{j,k,l})^* = \sqrt{[j,k,l]!}z_1^j z_2^k z_3^l, \qquad \text{if } N \geq 0 \quad \text{and with } j+k+l = N,$$
$$(\psi^N_{j,k,l})^* = \sqrt{[j,k,l]!}(z_1^j z_2^k z_3^l)^*, \qquad \text{if } N \leq 0 \quad \text{and with } i+j+k = -N.$$

Notice that we put an extra coefficient q^{-N}. This is needed for compatibility with the twist map in section (4.3).

The anti holomorphic part of this connection will be $\nabla^{\overline{\partial}}_N = q^{-N}\Psi_N^\dagger \overline{\partial}\Psi_N$. The curvature of $\nabla^{\overline{\partial}}_N$ can be computed as follows

$$(\nabla^{\overline{\partial}}_N)^2 = q^{-2N}\Psi_N^\dagger(\overline{\partial}P_N\overline{\partial}P_N)\Psi_N,$$

where $P_N := \Psi_N\Psi_N^\dagger$ is a projection map due to the fact that $\Psi_N^\dagger\Psi_N = 1$.

Proposition 4.2. *The connection* $\nabla^{\overline{\partial}}_N$ *is flat.*

Proof. We will prove this for $N \geq 0$ and a similar discussion will cover the case $N < 0$.

It suffices to show that

$$\Psi_N^\dagger\overline{\partial}P_N = \Psi_N^\dagger(P_N \triangleleft F_2F_1, P_N \triangleleft F_2)^t = 0.$$

The second component

$$\begin{aligned}\Psi_N^\dagger(P_N \triangleleft F_2) &= \Psi_N^\dagger((\Psi_N\Psi_N^\dagger)\triangleleft F_2)\\ &= \Psi_N^\dagger\{(\Psi_N\triangleleft F_2)(\Psi_N^\dagger\triangleleft K_2) + (\Psi_N\triangleleft K_2^{-1})(\Psi_N^\dagger\triangleleft F_2)\}\\ &= 0\,.\end{aligned}$$

and this last equality is obtained by (see [5], section 6)

$$\Psi_N^\dagger \triangleleft F_2 = 0\,, \quad \Psi_N^\dagger(\Psi_N\triangleleft F_2) = 0\,. \tag{16}$$

Similar computation shows that $\Psi_N^\dagger(P_N \triangleleft F_2F_1)$ also vanishes. For this the following identity is needed.

$$\Psi_N^\dagger(\Psi_N \triangleleft F_2F_1) = 0. \tag{17}$$

Hence $(\nabla_N^{\overline{\partial}})^2 = 0$. □

Alternatively, as it was kindly pointed out to us by Francesco D'Andrea, using Lemma 6.1 in [5], the full connection (holomorphic + antiholomorphic part) has curvature of type $(1,1)$. This implies that the square of the holomorphic and antiholomorphic part is zero.

Proposition (4.2) verifies that the operator $\nabla_N^{\overline{\partial}}$ satisfies the condition of holomorphic structure as given in the definition (2.3).

Flatness of $\nabla_N^{\overline{\partial}}$ gives the following complex of vector spaces

$$0 \to \mathcal{L}_N \to \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \mathcal{L}_N \to \Omega^{(0,2)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \mathcal{L}_N \to 0\,.$$

The zeroth cohomology group $H^0(\mathcal{L}_N, \nabla_N^{\overline{\partial}})$ of this complex is called the *space of holomorphic sections of* $\mathcal{L}_N$. The structure of this space is best described by the following theorem.

Theorem 4.1. *Let N be a positive integer. Then*

$$\begin{aligned}&(1) \qquad H^0(\mathcal{L}_N, \nabla_N^{\overline{\partial}}) \simeq \mathbb{C}^{\frac{(N+1)(N+2)}{2}}\\ &(2) \qquad H^0(\mathcal{L}_{-N}, \nabla_{-N}^{\overline{\partial}}) = 0.\end{aligned}$$

Proof. First we recall that

$$\nabla_N^{\overline{\partial}}\xi = q^{-N}\Psi_N^\dagger\overline{\partial}\Psi_N\xi = q^{-N}\Psi_N^\dagger((\Psi_N\xi)\triangleleft F_2F_1, (\Psi_N\xi)\triangleleft F_2)^t.$$

Using (16), (17) and the following identities

$$\Psi_N \triangleleft F_1 = 0, \quad \Psi_N\triangleleft K_1 = \Psi_N, \quad \Psi_N \triangleleft K_2 = q^{-N/2}\Psi_N, \tag{18}$$

we prove that $\nabla_N^{\overline{\partial}}\xi = 0$ is equivalent to the equations $\xi\triangleleft F_2 = 0$ and $\xi \triangleleft F_2F_1 = 0$.

First we compute the second component of $\nabla_N^{\overline{\partial}}\xi$.

$$\begin{aligned} q^{-N}\Psi_N^\dagger((\Psi_N\xi)\triangleleft F_2) &= q^{-N}\Psi_N^\dagger\{(\Psi_N\triangleleft F_2)(\xi\triangleleft K_2)+(\Psi_N\triangleleft K_2^{-1})(\xi\triangleleft F_2)\} \\ &= q^{-N/2}\xi\triangleleft F_2. \end{aligned}$$

In addition to (16) and (18), here we have used $\Psi_N^\dagger\Psi_N=1$. In a similar manner, one can show that the first component is

$$\begin{aligned} &q^{-N}\Psi_N^\dagger((\Psi_N\xi)\triangleleft F_2F_1) \\ &\qquad= q^{-N}\Psi_N^\dagger\{(\Psi_N\triangleleft F_2)(\xi\triangleleft K_2)+(\Psi_N\triangleleft K_2^{-1})(\xi\triangleleft F_2)\}\triangleleft F_1 \\ &\qquad= q^{-N}\Psi_N^\dagger\{q^{N/2}(\Psi_N\triangleleft F_2)\xi+q^{N/2}\Psi_N(\xi\triangleleft F_2)\}\triangleleft F_1 \\ &\qquad= q^{-N/2}\Psi_N^\dagger\{(\Psi_N\triangleleft F_2F_1)(\xi\triangleleft K_1)+(\Psi_N\triangleleft F_2K_1^{-1})(\xi\triangleleft F_1) \\ &\qquad+(\Psi_N\triangleleft F_1)(\xi\triangleleft F_2K_1)+(\Psi_N\triangleleft K_1^{-1})(\xi\triangleleft F_2F_1)\} \\ &\qquad= q^{-N/2}\xi\triangleleft F_2F_1. \end{aligned}$$

Let $N\geq 0$. In this case, a basis element of $\mathcal{L}_N$ is of the form $t(n,n+N)^{\underline{0}}_{\underline{j}}$. Similar computation to the proof of proposition 4.1, using (14), (8) and (9), shows that $t(n,n+N)^{\underline{0}}_{\underline{j}}\triangleleft F_2=\gamma_n t(n,n+N)^{1,0,-\frac{1}{2}}_{\underline{j}}$, where $\gamma_n=A_{0,0}=(\frac{[n][n+N+2]}{[2]})^{1/2}$. If $\xi\in\mathcal{L}_N$, then ξ can be written as $\sum_{n,\underline{j}}\lambda_{n,\underline{j}}t(n,n+N)^{\underline{0}}_{\underline{j}}$. So $\xi\triangleleft F_2=\sum\lambda_{n,\underline{j}}\gamma_n t(n,n+N)^{1,0,-1/2}_{\underline{j}}$. Since $\gamma_n=0$ iff $n=0$, $\xi\triangleleft F_2=0$ implies that the set $\{t(0,N)^{\underline{0}}_{\underline{j}}\}$ will form a basis for the space of $\mathrm{Ker}\nabla_N^{\overline{\partial}}$. Remembering that by (7), the indices are restricted by $j_1=0, j_2=0,...,N$, and $j_2/2-|m|\in\mathbb{N}$, we will find that $\dim\,\mathrm{Ker}\,\nabla_N^{\overline{\partial}}=\frac{(N+1)(N+2)}{2}$.

When N is a negative integer, γ_n will be $(\frac{[n-N][n+2]}{[2]})^{1/2}$ which is nonzero. So $\dim\,\mathrm{Ker}\,\nabla_N^{\overline{\partial}}=0$.

$\square$

4.3 Bimodule connections

There exists a $\mathcal{A}(\mathbb{C}P_q^2)$-bimodules isomorphism $\sigma:\Omega^{(0,1)}\otimes_{\mathcal{A}(\mathbb{C}P_q^2)}\mathcal{L}_N\to\mathcal{L}_N\otimes_{\mathcal{A}(\mathbb{C}P_q^2)}\Omega^{(0,1)}$ which acts as

$$\sigma(\omega\otimes\xi)=q^{-N}\xi'\otimes\omega',$$

such that both elements $\omega\otimes\xi$ and $\xi'\otimes\omega'$ in $\mathcal{A}(SU_q(3))^2$, after multiplication are the same. We try to illustrate this in the case of $N=1$. More precisely let us define the maps ϕ_1 and ϕ_2 as follows:

$$\begin{aligned} &\phi_1:\Omega^{(0,1)}\otimes_{\mathcal{A}(\mathbb{C}P_q^2)}\mathcal{L}_1\to\mathcal{A}(SU_q(3))^2, \\ &\phi_1((v_+,v_-)^t\otimes\xi)=q^{\frac{1}{2}}(v_+\xi,v_-\xi)^t, \end{aligned}$$

and

$$\phi_2 : \mathcal{L}_1 \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \Omega^{(0,1)} \to \mathcal{A}(SU_q(3))^2,$$
$$\phi_2(\xi \otimes (v_+, v_-)^t) = q^{-\frac{1}{2}}(\xi v_+, \xi v_-)^t.$$

We will prove that Im ϕ_1=Im ϕ_2. Therefore $\sigma = \phi_1^{-1}\phi_2$ gives an isomorphism from $\mathcal{L}_1 \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \Omega^{(0,1)}$ to $\Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} \mathcal{L}_1$ which is coming from the multiplication map. Let us first recall that as a $*$-algebra $\mathcal{A}(\mathbb{C}P_q^2)$ is generated by elements $p_{jk} = z_j^* z_k = (u_j^3)^* u_k^3$.

Lemma 4.1. *With above notation $Im\,\phi_1 = Im\,\phi_2$.*

Proof. case1. $\alpha \in$ Im ϕ_2 is a basis element.

$$\alpha = \phi_2(t(n,n+1)^{0}_{\underline{i}} \otimes p_{rs}\overline{\partial} p_{jk}) = q^{-1/2} t(n,n+1)^{0}_{\underline{i}} p_{rs} \begin{pmatrix} -q^{-3/2}(u_j^1)^* \\ q^{-1/2}(u_j^2)^* \end{pmatrix} u_k^3$$
$$= q^{-1/2} \begin{pmatrix} -q^{-3/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(u_j^1)^* \\ q^{-1/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(u_j^2)^* \end{pmatrix} u_k^3 = q^{-1}\phi_1(T_{\underline{i}rsj} \otimes u_k^3),$$

where

$$T_{\underline{i}rsj} = (-q^{-3/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(u_j^1)^*, q^{-1/2} p_{rs} t(n,n+1)^{0}_{\underline{i}} (u_j^2)^*)^t.$$

Since $u_k^3 \in \mathcal{L}_1$, it is enough to prove that $T_{\underline{i}rsj} \in \Omega^{(0,1)}$. In order to do so, we need to show that the pair (v_+, v_-) defined as below, satisfies the properties given in (15).

$$(v_+, v_-)^t = (-q^{-3/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(u_j^1)^*,\ q^{-1/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(u_j^2)^*)^t.$$

We will check $(v_+, v_-) \triangleleft E_1 = (v_-, 0)$.

$$\begin{aligned} v_+ \triangleleft E_1 &= -q^{-3/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(u_j^1)^* \triangleleft E_1 \\ &= -q^{-3/2}\{(t(n,n+1)^{0}_{\underline{i}} \triangleleft E_1)((p_{rs}(u_j^1)^*) \triangleleft K_1) \\ &+ (t(n,n+1)^{0}_{\underline{i}} \triangleleft K_1^{-1})((p_{rs}(u_j^1)^*) \triangleleft E_1)\} \\ &= -q^{-3/2} t(n,n+1)^{0}_{\underline{i}}\{(p_{rs} \triangleleft E_1)((u_j^1)^* \triangleleft K_1) \\ &+ (p_{rs} \triangleleft K_1^{-1})((u_j^1)^* \triangleleft E_1)\} \\ &= -q^{-3/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(-q)(u_j^2)^* \\ &= q^{-1/2} t(n,n+1)^{0}_{\underline{i}} p_{rs}(u_j^2)^* \\ &= v_-. \end{aligned}$$

Here we have used the following identities which are obtained from (8), (12) and (14).

$$\begin{aligned} &t(n,n+1)^{0}_{\underline{i}} \triangleleft K_1 = t(n,n+1)^{0}_{\underline{i}}, && t(n,n+1)^{0}_{\underline{i}} \triangleleft E_1 = 0 \\ &p_{ij} \triangleleft E_1 = 0, && (u_j^1)^* \triangleleft K_1 = q^{1/2}(u_j^1)^*, \\ &p_{ij} \triangleleft K_1 = p_{ij}, && (u_j^1)^* \triangleleft E_1 = (-q)(u_j^2)^*. \end{aligned}$$

Similarly

$$\begin{aligned}
v_- \triangleleft E_1 &= q^{-1/2} t(n,n+1)^{0}_{\underline{i}} p_{rs} (u^2_j)^* \triangleleft E_1 \\
&= q^{-1/2} \{ (t(n,n+1)^{0}_{\underline{i}} \triangleleft E_1)((p_{rs}(u^2_j)^*) \triangleleft K_1) \\
&+ (t(n,n+1)^{0}_{\underline{i}} \triangleleft K_1^{-1})((p_{rs}(u^2_j)^*) \triangleleft E_1) \} \\
&= q^{-1/2} t(n,n+1)^{0}_{\underline{i}} \{ (p_{rs} \triangleleft E_1)((u^2_j)^* \triangleleft K_1) \\
&+ (p_{rs} \triangleleft K_1^{-1})((u^2_j)^* \triangleleft E_1) \} \\
&= 0.
\end{aligned}$$

Two more identities which have been used above, are

$$(u^2_j)^* \triangleleft K_1 = q^{-1/2} (u^1_j)^*, \qquad (u^2_j)^* \triangleleft E_1 = 0.$$

The case $(v_+, v_-) \triangleleft F_1 = (0, v_+)$ is similar and the other two cases $(v_+, v_-) \triangleleft K_1 = (q^{1/2} v_+, q^{-1/2} v_-)$ and $(v_+, v_-) \triangleleft K_1 K_2^2 = q^{3/2}(v_+, v_-)$ are straightforward, but the following relations are needed.

$$\begin{aligned}
&t(n,n+1)^{0}_{\underline{i}} \triangleleft K_2 = q^{1/2} t(n,n+1)^{0}_{\underline{i}}, && t(n,n+1)^{0}_{\underline{i}} \triangleleft F_1 = 0, \\
&(u^1_j)^* \triangleleft K_2 = (u^1_j)^*, && (u^2_j)^* \triangleleft K_2 = q^{1/2} (u^2_j)^*, \\
&(u^1_j)^* \triangleleft F_1 = 0, && (u^2_j)^* \triangleleft F_1 = (-q)^{-1} (u^1_j)^*, \\
&p_{ij} \triangleleft K_2 = p_{ij}, && p_{ij} \triangleleft F_1 = 0.
\end{aligned}$$

Case2. $\alpha \in \mathrm{Im}\ \phi_2$ is a general element.

$$\begin{aligned}
\alpha &= \phi_2 \Big(\sum_{n,\underline{i}} c_{n\underline{i}} t(n,n+1)^{0}_{\underline{i}} \otimes \sum_{r,s,j,k} d_{rsjk} p_{rs} \overline{\partial} p_{jk} \Big) \\
&= q^{-1/2} \sum_{n,\underline{i},r,s,j,k} c_{n\underline{i}} t(n,n+1)^{0}_{\underline{i}} d_{rsjk} p_{rs} \begin{pmatrix} -q^{-3/2}(u^1_j)^* \\ q^{-1/2}(u^2_j)^* \end{pmatrix} u^3_k \\
&= q^{-1} \phi_1 \Big(\sum_k \Big\{ \sum_{\underline{i},r,s,j} c_{\underline{i}} d_{rsjk} t(n,n+1)^{0}_{\underline{i}} p_{rs} \begin{pmatrix} -q^{-3/2}(u^1_j)^* \\ q^{-1/2}(u^2_j)^* \end{pmatrix} \Big\} \otimes u^3_k \Big) \\
&= q^{-1} \phi_1 \Big(\sum_k A_k \otimes u^3_k \Big),
\end{aligned}$$

where

$$A_k = \sum_{n,\underline{i},r,s,j} c_{n\underline{i}} d_{rsjk} t(n,n+1)^{0}_{\underline{i}} p_{rs} \begin{pmatrix} q^{-3/2}(u^1_j)^* \\ q^{-1/2}(u^2_j)^* \end{pmatrix} \in \Omega^{(0,1)}.$$

The proof for $Im\ \phi_2 \subset Im\ \phi_1$ is similar.

□

In general the maps ϕ_1 and ϕ_2 will be defined as

$$\phi_1 : \Omega^{(0,1)} \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \mathcal{L}_N \to \mathcal{A}(SU_q(3))^2,$$
$$\phi_1((v_+, v_-)^t \otimes \xi) = q^{\frac{N}{2}} (v_+ \xi, v_- \xi)^t,$$

and

$$\phi_2 : \mathcal{L}_N \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \Omega^{(0,1)} \to \mathcal{A}(SU_q(3))^2,$$
$$\phi_2(\xi \otimes (v_+, v_-)) = q^{-\frac{N}{2}}(\xi v_+, \xi v_-).$$

Now, we prove that $\nabla^{\overline{\partial}}_N$ has the right σ-twisted Leibniz property with respect to the map $\sigma = \phi_1^{-1}\phi_2$.

Proposition 4.3. *Taking σ as above, the following holds*

$$\nabla^{\overline{\partial}}_N(\xi a) = (\nabla^{\overline{\partial}}_N \xi)a + \sigma(\xi \otimes \overline{\partial} a), \quad \forall a \in \mathcal{A}(\mathbb{C}P^2_q), \forall \xi \in \mathcal{L}_N. \tag{19}$$

Proof. By (16), (18) and the fact that $\xi \triangleleft K_2 = q^{N/2}\xi$, we compute the second component of the left hand side as follows

$$\begin{aligned} &q^{-N}\Psi^\dagger_N((\Psi_N \xi a) \triangleleft F_2) \\ &\qquad = q^{-N}\Psi^\dagger_N\{(\Psi_N \triangleleft F_2)((\xi a) \triangleleft K_2) + (\Psi_N \triangleleft K_2^{-1})((\xi a) \triangleleft F_2)\} \\ &\qquad = q^{-N/2}(\xi \triangleleft F_2)a + q^{-N}\xi(a \triangleleft F_2). \end{aligned}$$

(Note that this actually is $\phi_1 \nabla^{\overline{\partial}}_N$.) For the second component of the right hand side we will get

$$q^{-N/2}(\xi \triangleleft F_2)a + \sigma(\xi \otimes a \triangleleft F_2).$$

The previous lemma says that q^{-N} will appear after acting σ on the second term. It can be seen that ϕ_1 of both sides coincides. Computation for the second component will be similar. ☐

Now we will come up to the analog of proposition 3.8 of ([7]).

Proposition 4.4. *The tensor product connection $\nabla^{\overline{\partial}}_N \otimes 1 + (\sigma \otimes 1)(1 \otimes \nabla^{\overline{\partial}}_M)$ coincides with the holomorphic structure on $\mathcal{L}_N \otimes_{\mathcal{A}(\mathbb{C}P^1_q)} \mathcal{L}_M$ when identified with $\mathcal{L}_{N+M}$.*

Proof.

$$\begin{aligned} &\nabla^{\overline{\partial}}_{N+M}(\xi_1\xi_2) \\ &= q^{-(N+M)}\Psi^\dagger_{N+M}\overline{\partial}\Psi_{N+M}(\xi_1\xi_2) \\ &= q^{-(N+M)}\Psi^\dagger_{N+M}\begin{pmatrix} (\Psi_{N+M}\xi_1\xi_2) \triangleleft F_2F_1 \\ (\Psi_{N+M}\xi_1\xi_2) \triangleleft F_2 \end{pmatrix} \\ &= q^{-(N+M)}\Psi^\dagger_{N+M}\begin{pmatrix} \{(\Psi_{N+M} \triangleleft F_2)((\xi_1\xi_2) \triangleleft K_2)\} \triangleleft F_1 \\ (\Psi_{N+M} \triangleleft F_2)((\xi_1\xi_2) \triangleleft K_2) \end{pmatrix} \\ &+q^{-(N+M)}\Psi^\dagger_{N+M}\begin{pmatrix} \{(\Psi_{N+M} \triangleleft K_2^{-1})((\xi_1\xi_2) \triangleleft F_2)\} \triangleleft F_1 \\ (\Psi_{N+M} \triangleleft K_2^{-1})((\xi_1\xi_2) \triangleleft F_2) \end{pmatrix} \\ &= q^{-\frac{N+M}{2}}\begin{pmatrix} (\xi_1\xi_2) \triangleleft F_2F_1 \\ (\xi_1\xi_2) \triangleleft F_2 \end{pmatrix} \\ &= q^{-\frac{N}{2}}\begin{pmatrix} \{(\xi_1 \triangleleft F_2)\xi_2 + (q^{-N-M/2}\xi_1(\xi_2 \triangleleft F_2)\} \triangleleft F_1 \\ (\xi_1 \triangleleft F_2)\xi_2 + q^{-N-M/2}\xi_1(\xi_2 \triangleleft F_2) \end{pmatrix}. \end{aligned}$$

Besides (16) and (17), we also applied the identities $\xi_i \triangleleft K_1 = 0$, $\xi_i \triangleleft F_1 = 0$.

On the other hand

$$((\nabla^{\overline{\partial}}_N \otimes 1) + (\sigma \otimes 1)(1 \otimes \nabla^{\overline{\partial}}_M))(\xi_1 \otimes \xi_2) =$$
$$q^{-N/2}\begin{pmatrix} \xi_1 \triangleleft F_2F_1 \\ \xi_1 \triangleleft F_2 \end{pmatrix} \otimes \xi_2 + (\sigma \otimes 1)(\xi_1 \otimes q^{-M/2}\begin{pmatrix} \xi_2 \triangleleft F_2F_1 \\ \xi_2 \triangleleft F_2 \end{pmatrix}).$$

Interpreting this expression as an element of $\Omega^{(0,1)} \otimes \mathcal{L}_{N+M}$, after applying the map σ, which gives us q^{-N} on the second summand, we will get the same result. $\square$

Thanks to proposition (4.4), the space $R := \bigoplus H^0(\mathcal{L}_N, \nabla^{\overline{\partial}}_N)$ has a ring structure under the natural tensor product of bimodules. In the following, we identify the quantum homogeneous coordinate ring R with a twisted polynomial algebra in three variables

Theorem 4.2. *We have the algebra isomorphism*

$$R := \bigoplus_{N \geq 0} H^0(\mathcal{L}_N, \nabla^{\overline{\partial}}_N) \simeq \frac{\mathbb{C}\langle z_1, z_2, z_3\rangle}{\langle\, z_iz_j - qz_jz_i : 1 \leq i < j \leq 3\,\rangle}$$

Proof. The ring structure on R is coming from the tensor product $\mathcal{L}_{N_1} \otimes_{\mathcal{A}(\mathbb{C}P^2_q)} \mathcal{L}_{N_2} \simeq \mathcal{L}_{N_1+N_2}$. The following discussion shows that $H^0(\mathcal{L}_1, \nabla^{\overline{\partial}}_1) = \mathbb{C}z_1 \oplus \mathbb{C}z_2 \oplus \mathbb{C}z_3$. In order to do this, we will give an explicit formula for the basis elements of $H^0(\mathcal{L}_N, \nabla^{\overline{\partial}}_N)$, $t(0,N)^{\underline{0}}_{\underline{j}}$.

Let us look at the computation more closely. Using (13), we will see that

$$t(0,N)^{\underline{0}}_{\underline{j}} = [j_2+1]\sqrt{\frac{[\frac{j_2}{2}+m]![N-j_2]!}{[\frac{j_2}{2}-m]![N]!}}\,F_1^{1/2j_2-m}F_2^{j_2} \triangleright z_3^N. \tag{20}$$

By induction it is not difficult to prove that $F_2 \triangleright z_3^N = q^{-\frac{N-1}{2}}[N]z_2z_3^{N-1}$. Therefore

$$\begin{aligned} F_2^j \triangleright z_3^N &= q^{-\frac{N-1}{2}-\frac{N-2}{2}-\ldots-\frac{N-j}{2}+\frac{1}{2}+\ldots+\frac{j-1}{2}}[N]\ldots[N-j+1]z_2^jz_3^{N-j} \\ &= q^{\frac{j^2}{2}-\frac{jN}{2}}\frac{[N]!}{[N-j]!}z_2^jz_3^{N-j}. \end{aligned}$$

The same method gives

$$F_1^r \triangleright z^j = q^{\frac{r^2}{2}-\frac{rj}{2}}\frac{[j]!}{[j-r]!}z_1^rz_2^{j-r}.$$

So

$$F_1^rF_2^j \triangleright z_3^N = q^{\frac{j^2}{2}-\frac{jN}{2}+\frac{r^2}{2}-\frac{rj}{2}}\frac{[N]!}{[N-j]!}\frac{[j]!}{[j-r]!}z_1^rz_2^{j-r}z_3^{N-j}.$$

Replacing j with j_2 and r with $1/2j_2 - m$, we will have

$$t(0,N)^{\underline{0}}_{\underline{j}} = [j_2+1]!\sqrt{\frac{[N]!}{[\frac{j_2}{2}-m]![\frac{j_2}{2}+m]![N-j_2]!}}q^{\alpha}z_1^{1/2j_2-m}z_2^{1/2j_2+m}z_3^{N-j_2},$$

where $\alpha = -\frac{j_2N}{2} - (\frac{j_2}{2}-m)\frac{j_2}{2} + \frac{j_2^2}{2} + \frac{1}{2}(\frac{j_2}{2}-m)^2$.

In the case $N=1$, $t(0,1)^{\underline{0}}_{0,1,-\frac{1}{2}} = [2]z_1$, $t(0,1)^{\underline{0}}_{0,1,\frac{1}{2}} = q[2]z_2$ and $t(0,1)^{\underline{0}}_{\underline{0}} = z_3$. Now the isomorphism follows from the identities $z_i \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} z_j - q z_j \otimes_{\mathcal{A}(\mathbb{C}P_q^2)} z_i = 0$ in $\mathcal{L}_2$, which can easily be seen.

□

5 The C^*-algebras $C(SU_q(3))$ and $C(\mathbb{C}P_q^2)$

In this section we extend the results of Proposition (4.1) and Theorem (4.1) which are stated for polynomial functions and polynomial sections to L^2-functions and sections, respectively.

Let $C(SU_q(3))$ denotes the C^* completion of $\mathcal{A}(SU_q(3))$, i.e. the universal C^*-algebra generated by the elements u^i_j subject to the relations given in section 3.2. There exists a unique left invariant normalized Haar state on this compact quantum group denoted by h. The functional h is faithful and it also has a twisted tracial property which will be considered in the next section. If we denote the Hilbert space of completion of $\mathcal{A}(SU_q(3))$ with respect to the inner product $\langle a, b\rangle := h(a^*b)$ by $L^2(SU_q(3))$. Since the Haar state on the C^*-algebra $C(SU_q(3))$ is faithfull [9], the GNS map $\eta : C(SU_q(3)) \to L^2(SU_q(3))$ will be injective. An orthogonal basis of $L^2(SU_q(3))$, would be $\eta(t(n_1,n_2)^{\underline{l}}_{\underline{j}})$.

Using the fact that $L^2(SU_q(3))$ is the completion of $\bigoplus_{(n_1,n_2)\in\mathbb{N}^2} V(n_1,n_2)\otimes V(n_1,n_2)$, the action of $U_q(\mathfrak{su}(3))$ naturally rises to an action on $L^2(SU_q(3))$. The invariant subalgebra of $C(SU_q(3))$ under the action of $U_q(\mathfrak{u}(2))$ is by definition the C^*-algebra $C(\mathbb{C}P_q^2)$. By the invariance property of the Haar state, the GNS map restricts to an injective map $C(\mathbb{C}P_q^2) \to L^2(SU_q(3))^{U_q(\mathfrak{u}(2))}$. The space of continuous sections and L^2-sections can be defined as well.

$$\begin{aligned}
\Gamma(\mathcal{L}_N) :&= \{\xi \in C(SU_q(3)) | \xi \triangleleft k = \epsilon(k)\xi,\ \xi \triangleleft K_1K_2^2 = q^N\xi, \quad \forall k \in U_q(\mathfrak{u}(2))\} \\
L^2(\mathcal{L}_N) :&= \{\xi \in L^2(SU_q(3)) | \xi \triangleleft k = \epsilon(k)\xi,\ \xi \triangleleft K_1K_2^2 = q^N\xi, \quad \forall k \in U_q(\mathfrak{u}(2))\} \\
&= \mathrm{Span}\{t(n,n+N)^{\underline{0}}_{\underline{j}} |\ n \in \mathbb{N},\ \ \underline{j} \text{ satisfies } (7)\}^{closure}
\end{aligned}$$

Note the the last equality is for $N \geq 0$. For $N < 0$, basis elements are of the form $t(n-N,n)^{\underline{0}}_{\underline{j}}$.

The operator $Z = \triangleleft(F_2F_1, F_2)$ is unbouded on $L^2(SU_q(3))$, so we have to specify the domain of this operator.

$$\mathrm{Dom}(Z) := \{a \in L^2(SU_q(3)) |\, (a \triangleleft F_2F_1, a \triangleleft F_2) \in L^2(SU_q(3)^2)\}.$$

Now the Proposition 4.1 can easily be generalized to the following proposition.

Proposition 5.1. *The Kernel of the map Z restricted to $L^2(\mathbb{C}P_q^2)$ is $\mathbb{C}$.*

Proof. Since any element of $L^2(\mathbb{C}P_q^2)$ is a L^2-linear combination of the elements $t(n,n)^{\underline{0}}_{\underline{j}}$, proof is exactly like Proposition 4.1. □

Let us define $\mathrm{Dom}(\overline{\partial}) := \{a \in C(\mathbb{C}P_q^2) | \, ||\overline{\partial}a|| < \infty\}$. The above statement could pass to continuous functions as follows.

Corollary 5.1. *There is no non-constant holomorphic function in $C(\mathbb{C}P_q^2)$.*

With a similar discussion, the analog of 4.1 continues to hold if we work with L^2-sections of $\mathcal{L}_N$. We give the statement of the theorem and leave its similar proof to the reader.

Theorem 5.1. *Let N be a positive integer. Then*

$$H^0(L^2(\mathcal{L}_N), \nabla_N^{\overline{\partial}}) \simeq \mathbb{C}^{\frac{(N+1)(N+2)}{2}}, \tag{1}$$

$$H^0(L^2(\mathcal{L}_{-N}), \nabla_{-N}^{\overline{\partial}}) = 0. \tag{2}$$

We note that our approach here as well as in [7], is somehow the opposite of the approach adopted in [1, 2] to noncommutative projective spaces. We started with a C^*-algebra defined as the quantum homogeneous space of the quantum group $SU_q(3)$ and its natural line bundles, and endowed them with holomorphic structures. The quantum homogeneous coordinate ring is then defined as the algebra of holomorphic sections of these line bundles. This ring coincides with the twisted homogeneous ring associated in [1, 2] to the line bundle $\mathcal{O}(1)$ under a suitable twist.

6 Existence of a twisted positive Hochschild 4-cocycle on $\mathbb{C}P_q^2$

In [3], Section VI.2, Connes shows that extremal positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a holomorphic structure on the surface. There is a similar result for holomorphic structures on the noncommutative two torus (cf. *Loc cit.*). In particular the positive Hocshchild cocycle is defined via the holomorphic structure and represents the fundamental cyclic cocycle. In [7] a notion of twisted positive Hochschild cocycle is introduced and a similar result is proved for the holomorphic structure of $\mathbb{C}P_q^1$. Although the corresponding problem of characterizing holomorphic structures on higher dimensional (commutative or noncommutative) manifolds via positive Hochschild cocycles is still open, nevertheless these results suggest regarding (twisted) positive Hochschild cocycles as a possible framework for holomorphic noncommutative structures. In this section we prove an analogous result for $\mathbb{C}P_q^2$.

First we recall the notion of twisted Hochschild and cyclic cohomologies. Let $\mathcal{A}$ be an algebra and σ an automorphism of $\mathcal{A}$. For each $n \geq 0$, $C^n(\mathcal{A}) :=$

$\mathrm{Hom}(\mathcal{A}^{\otimes(n+1)}, \mathbb{C})$ is the space of *n-cochains* on $\mathcal{A}$. Define the space of *twisted Hochschild n-cochains* as $C^n_\sigma(\mathcal{A}) :=\mathrm{Ker}\{(1-\lambda_\sigma^{n+1}) : C^n(\mathcal{A}) \to C^n(\mathcal{A})\}$, where the *twisted cyclic* map $\lambda_\sigma : C^n(\mathcal{A}) \to C^n(\mathcal{A})$ is defined as

$$(\lambda_\sigma\phi)(a_0, a_1, ..., a_n) = (-1)^n \phi(\sigma(a_n), a_0, a_1, ..., a_{n-1}).$$

The *twisted Hochschild coboundary* map $b_\sigma : C^n(\mathcal{A}) \to C^{n+1}(\mathcal{A})$ is given by

$$\begin{aligned} b_\sigma\phi(a_0, a_1, ..., a_{n+1}) = &\sum_{i=0}^{n} (-1)^i \phi(a_0, ..., a_i a_{i+1}, ..., a_{n+1}) \\ &+ (-1)^{n+1}\phi(\sigma(a_{n+1})a_0, ..., a_n). \end{aligned}$$

The cohomology of the complex $(C^*_\sigma(\mathcal{A}), b_\sigma)$ is called the *twisted Hochschild cohomology* of $\mathcal{A}$. We also need the notion of *twisted cyclic cohomology* of $\mathcal{A}$. It is by definition the cohomology of the complex $(C^*_{\sigma,\lambda}(\mathcal{A}), b_\sigma)$, where

$$C^n_{\sigma,\lambda} := Ker\{(1-\lambda) : C^n_\sigma(\mathcal{A}) \to C^{n+1}_\sigma(\mathcal{A})\}.$$

Now we come back to the case of our interest, that is $\mathbb{C}P^2_q$. Let τ be the fundamental class on $\mathbb{C}P^2_q$ defined as in [5] by

$$\tau(a_0, a_1, a_2, a_3, a_4) := -\int_h a_0 \, \mathrm{d}a_1 \, \mathrm{d}a_2 \, \mathrm{d}a_3 \, \mathrm{d}a_4 \,, \quad \forall a_0, a_1, ..., a_4 \in \mathcal{A}(\mathbb{C}P^2_q). \tag{21}$$

Here h stands for the Haar state functional of the quantum group $\mathcal{A}(SU_q(3))$ which has a twisted tracial property $h(xy) = h(\sigma(y)x)$. Here the algebra automorphism σ is defined by

$$\sigma : \mathcal{A}(SU_q(3)) \to \mathcal{A}(SU_q(3)), \quad \sigma(x) = K \triangleright x \triangleleft K.$$

where $K = (K_1K_2)^{-4}$. The map σ, restricted to the algebra $\mathcal{A}(\mathbb{C}P^2_q)$ is given by $\sigma(x) = K \triangleright x$. Non-triviality of τ has been shown in [5]. Now we recall the definition of a twisted positive Hochschild cocycle as given in [7].

Definition 6.1. *A twisted Hochschild 2n-cocycle ϕ on a $*$-algebra $\mathcal{A}$ is said to be twisted positive if the following map defines a positive sesquilinear form on the vector space $\mathcal{A}^{\otimes(n+1)}$:*

$$\langle a_0 \otimes a_1 \otimes ... \otimes a_n, b_0 \otimes b_1 \otimes ... \otimes b_n\rangle = \phi(\sigma(b_n^*)a_0, a_1, ..., a_n, b_n^*, ..., b_1^*).$$

We would like to define a twisted Hochschild cocycle φ which is cohomologous to τ and it is positive. For simplicity, we introduce first the maps φ_i, for $i = 1, 2$ as follows

$$\begin{aligned} \varphi_1(a_0, a_1, a_2, a_3, a_4) &= -3\int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4, \\ \varphi_2(a_0, a_1, a_2, a_3, a_4) &= -3\int_h a_0 \overline{\partial} a_1 \overline{\partial} a_2 \partial a_3 \partial a_4. \end{aligned} \tag{22}$$

Now we define $\varphi \in C^4(\mathcal{A}(\mathbb{C}P_q^2))$ by

$$\varphi := \varphi_1 + \varphi_2 \,. \tag{23}$$

We will need the following simple lemma for future computations.

Lemma 6.1. *For any $a_0, a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}(\mathbb{C}P_q^2)$ the following identities hold:*

$$\int_h a_0(\partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4) a_5 = \int_h \sigma(a_5) a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4,$$
$$\int_h a_0(\overline{\partial} a_1 \overline{\partial} a_2 \partial a_3 \partial a_4) a_5 = \int_h \sigma(a_5) a_0 \overline{\partial} a_1 \overline{\partial} a_2 \partial a_3 \partial a_4.$$

Proof. We give the proof of the first one. The proof for the second equality will be similar. The space of $\Omega^{(2,2)}$ is a rank one free $\mathcal{A}(\mathbb{C}P_q^2)$-module. Let ω be the central basis element for the space of $\Omega^{(2,2)}$ and let $\partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 = x\omega$. Then

$$\begin{aligned}
\int_h a_0(\partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4) a_5 - \int_h \sigma(a_5) a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 &= \int_h (a_0 x \omega a_5 - \sigma(a_5) a_0 x \omega) \\
&= \int_h (a_0 x a_5 \omega - \sigma(a_5) a_0 x \omega) \\
&= h(a_0 x a_5 - \sigma(a_5) a_0 x) = 0.
\end{aligned}$$

The last equality comes from the twisted property of the Haar state. □

Proposition 6.1. *The functional φ defined by formula (23), is a twisted positive Hochschild 4-cocycle.*

Proof. We first verify the twisted cocycle property. In order to do so, we consider this property for each φ_i. We will prove the statement for φ_1. The proof for φ_2 is similar.

$$\begin{aligned}
&\varphi_1(\sigma(a_0), \sigma(a_1), \sigma(a_2), \sigma(a_3), \sigma(a_4)) \\
&= -3 \int_h \sigma(a_0) \partial \sigma(a_1) \partial \sigma(a_2) \overline{\partial} \sigma(a_3) \overline{\partial} \sigma(a_4) \\
&= -3 \int_h (K \triangleright a_0)(K \triangleright \partial a_1)(K \triangleright \partial a_2)(K \triangleright \overline{\partial} a_3)(K \triangleright \overline{\partial} a_4) \\
&= -3 \int_h K \triangleright (a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4) = -3\, \epsilon(K) \int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 \\
&= \varphi_1(a_0, a_1, a_2, a_3, a_4).
\end{aligned}$$

Now let us prove that $b_\sigma \varphi = 0$. Again we just prove for φ_1 and leave the similar proof of the other one.

$$\begin{aligned}
b_\sigma \varphi_1(a_0, a_1, a_2, a_3, a_4, a_5) &= \varphi_1(a_0 a_1, a_2, a_3, a_4, a_5) - \varphi_1(a_0, a_1 a_2, a_3, a_4, a_5) \\
&\quad + \varphi_1(a_0, a_1, a_2 a_3, a_4, a_5) - \varphi_1(a_0, a_1, a_2, a_3 a_4, a_5) \\
&\quad + \varphi_1(a_0, a_1, a_2, a_3, a_4 a_5) - \varphi_1(\sigma(a_5) a_0, a_1, a_2, a_3, a_4)
\end{aligned}$$

Using (22), this equals to

$$\begin{aligned}
&-3\int_h a_0 a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 + 3\int_h a_0 \partial(a_1 a_2)\partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 \\
&-3\int_h a_0 \partial a_1 \partial(a_2 a_3)\overline{\partial} a_4 \overline{\partial} a_5 + 3\int_h a_0 \partial a_1 \partial a_2 \overline{\partial}(a_3 a_4)\overline{\partial} a_5 \\
&-3\int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial}(a_4 a_5) + 3\int_h \sigma(a_5) a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4.
\end{aligned}$$

Using the Leibniz property we get

$$b_\sigma \varphi_1(a_0, a_1, a_2, a_3, a_4, a_5) = -3\int_h (a_0 a_1 \partial a_2 \partial a_3 \overline{\partial} a_4 \overline{\partial} a_5 - \sigma(a_5) a_0 \partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4),$$

which is zero by the previous lemma.

Now we will show that all φ_1 and φ_2 are positive.

Positivity of φ_1:

$$\begin{aligned}
\varphi_1(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) &= -3\int_h \sigma(a_0^*) a_0 \partial a_1 \partial a_2 \overline{\partial} a_2^* \overline{\partial} a_1^* \\
&= -3\int_h a_0 \partial a_1 \partial a_2 \overline{\partial} a_2^* \overline{\partial} a_1^* a_0^* \\
&= 3\int_h (a_0 \partial a_1 \partial a_2)(a_0 \partial a_1 \partial a_2)^*.
\end{aligned}$$

One can take $\partial a_1 = (v_1, v_2)$ and $\partial a_2 = (w_1, w_2)$, then using the multiplication rule of type (1,0) forms (c.f. [5] Proposition A.1), we find that $(a_0 \partial a_1 \partial a_2)(a_0 \partial a_1 \partial a_2)^* = c_4^2 [2]^{-1} \mu\mu^*$, where $\mu = q^{1/2} a_0 v_1 w_2 - q^{-1/2} a_0 v_2 w_1$. Hence

$$\varphi_1(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) = h(3c_4^2 [2]^{-1} \mu\mu^*) \geq 0.$$

Positivity of φ_2:

$$\begin{aligned}
\varphi_2(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) &= -3\int_h \sigma(a_0^*) a_0 \overline{\partial} a_1 \overline{\partial} a_2 \partial a_2^* \partial a_1^* \\
&= -3\int_h a_0 \overline{\partial} a_1 \overline{\partial} a_2 \partial a_2^* \partial a_1^* a_0^* \\
&= 3\int_h (a_0 \overline{\partial} a_1 \overline{\partial} a_2)(a_0 \overline{\partial} a_1 \overline{\partial} a_2)^*.
\end{aligned}$$

Similar to the above discussion, one can take $\overline{\partial} a_1 = (v_1, v_2)$ and $\overline{\partial} a_2 = (w_1, w_2)$ and use the multiplication of type (0,1) forms to find that

$$\varphi_2(\sigma(a_0^*)a_0, a_1, a_2, a_2^*, a_1^*) = h(3c_0^2 [2]^{-1} \nu\nu^*) \geq 0,$$

where $\nu = q^{1/2} a_0 v_1 w_2 - q^{-1/2} a_0 v_2 w_1$. Here c_0 and c_4 are two real constants. This concludes the positivity of φ.

□

Now we want to show that the twisted Hochschild cocycle φ as defined by formula (23) and the twisted cyclic cocycle τ as in formula (21) are cohomologous. To this end, we need an appropriate twisted Hochschild cocycle ψ such that $\tau - \varphi = b_\sigma \psi$. Let ψ_i for , i=1,2,3,4 be defined by

$$\begin{aligned}
\psi_1(a_0, a_1, a_2, a_3) &= -\int_h a_0 \partial a_1 \overline{\partial} a_2 \partial \overline{\partial} a_3, \\
\psi_2(a_0, a_1, a_2, a_3) &= 2\int_h a_0 \partial a_1 \partial \overline{\partial} a_2 \overline{\partial} a_3, \\
\psi_3(a_0, a_1, a_2, a_3) &= 2\int_h a_0 \overline{\partial} a_1 \overline{\partial} \partial a_2 \partial a_3, \\
\psi_4(a_0, a_1, a_2, a_3) &= -\int_h a_0 \overline{\partial} a_1 \partial a_2 \overline{\partial} \partial a_3.
\end{aligned}$$

and let $\psi = \sum_{i=1}^4 \psi_i$. Then we will have the following result.

Proposition 6.2. *The twisted Hochschild cocycles τ and φ are cohomologous.*

Proof.

$$\begin{aligned}
b_\sigma \psi_1(a_0, a_1, a_2, a_3, a_4) = {} & \psi_1(a_0 a_1, a_2, a_3, a_4) - \psi_1(a_0, a_1 a_2, a_3, a_4) \\
& + \psi_1(a_0, a_1, a_2 a_3, a_4) - \psi_1(a_0, a_1, a_2, a_3 a_4) \\
& + \psi_1(\sigma(a_4) a_0, a_1, a_2, a_3)
\end{aligned}$$

which equals to

$$\begin{aligned}
-\int_h \{ & a_0 a_1 \partial a_2 \overline{\partial} a_3 \partial \overline{\partial} a_4 - a_0 \partial(a_1 a_2) \overline{\partial} a_3 \partial \overline{\partial} a_4 + a_0 \partial a_1 \overline{\partial}(a_2 a_3) \partial \overline{\partial} a_4 \\
& - a_0 \partial a_1 \overline{\partial} a_2 \partial \overline{\partial}(a_3 a_4) + \sigma(a_4) a_0 \partial a_1 \overline{\partial} a_2 \partial \overline{\partial} a_3 \}.
\end{aligned}$$

Applying the Leibniz rule, one can see that in the expanded form, all but two terms will cancel. That is

$$b_\sigma \psi_1 = \int_h a_0 (\partial a_1 \overline{\partial} a_2 \partial a_3 \overline{\partial} a_4 - \partial a_1 \overline{\partial} a_2 \overline{\partial} a_3 \partial a_4).$$

Similar computation for ψ_i, $i = 2, 3$ and 4 shows that

$$\begin{aligned}
b_\sigma \psi_2 &= 2\int_h a_0 (\partial a_1 \partial a_2 \overline{\partial} a_3 \overline{\partial} a_4 - \partial a_1 \overline{\partial} a_2 \partial a_3 \overline{\partial} a_4), \\
b_\sigma \psi_3 &= 2\int_h a_0 (\overline{\partial} a_1 \overline{\partial} a_2 \partial a_3 \partial a_4 - \overline{\partial} a_1 \partial a_2 \overline{\partial} a_3 \partial a_4), \\
b_\sigma \psi_4 &= \int_h a_0 (\overline{\partial} a_1 \partial a_2 \overline{\partial} a_3 \partial a_4 - \overline{\partial} a_1 \partial a_2 \partial a_3 \overline{\partial} a_4).
\end{aligned}$$

Therefore

$$\begin{aligned} b_\sigma\psi = 2\int_h & a_0(\partial a_1\partial a_2\overline{\partial} a_3\overline{\partial} a_4 + \overline{\partial} a_1\overline{\partial} a_2\partial a_3\partial a_4) \\ - \int_h & a_0(\overline{\partial} a_1\partial a_2\partial a_3\overline{\partial} a_4 + \partial a_1\overline{\partial} a_2\overline{\partial} a_3\partial a_4) \\ - \int_h & a_0(\overline{\partial} a_1\partial a_2\overline{\partial} a_3\partial a_4 + \partial a_1\overline{\partial} a_2\partial a_3\overline{\partial} a_4). \end{aligned} \tag{24}$$

Now from (21), (23) and (24), we can easily find that $\tau - \varphi = b_\sigma\psi$. $\square$

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Department of Mathematics, University of Western Ontario, London, Ontario, N6A5B7, Canada.
Email: masoud@uwo.ca

Department of Mathematics, University of Western Ontario, London, Ontario, N6A5B7, Canada.
Email: amotadel@uwo.ca